Proceedings of IDETC/CIE 2013
ASME 2013 International Design Engineering Technical Conferences
& Computers and Information in Engineering Conference
August 4-7, 2013, Portland, OR, USA

DETC2013-12257

Time-dependent Reliability of Dynamic Systems using Subset Simulation with Splitting over a Series of Correlated Time Intervals

Zhonglai Wang[1]
Zissimos P. Mourelatos*
Jing Li

Mechanical Engineering Department
Oakland University, Rochester, MI 48309, USA

Amandeep Singh
Igor Baseski

U.S. Army TARDEC,
Warren, MI 48397, USA

[1]Visiting scholar from University of Electronic Science and Technology of China, Chengdu, Sichuan 611731

ABSTRACT

Time-dependent reliability is the probability that a system will perform its intended function successfully for a specified time. Unless many and often unrealistic assumptions are made, the accuracy and efficiency of time-dependent reliability estimation are major issues which may limit its practicality. Monte Carlo simulation (MCS) is accurate and easy to use but it is computationally prohibitive for high dimensional, long duration, time-dependent (dynamic) systems with a low failure probability. This work addresses systems with random parameters excited by stochastic processes. Their response is calculated by time integrating a set of differential equations at discrete times. The limit state functions are therefore, explicit in time and depend on time-invariant random variables and time-dependent stochastic processes. We present an improved subset simulation with splitting approach by partitioning the original high dimensional random process into a series of correlated, short duration, low dimensional random processes. Subset simulation reduces the computational cost by introducing appropriate intermediate failure sub-domains to express the low failure probability as a product of larger conditional failure probabilities. Splitting is an efficient sampling method to estimate the conditional probabilities. The proposed subset simulation with splitting not only estimates the time-dependent probability of failure at a given time but also estimates the cumulative distribution function up to that time with approximately the same cost. A vibration example involving a vehicle on a stochastic road demonstrates the advantages of the proposed approach.

1. INTRODUCTION

Reliability is an important engineering requirement for consistently delivering acceptable product performance through time. As time progresses, the product may fail due to time-dependent operating conditions and material properties, component degradation, etc. The reliability degradation with time may increase the lifecycle cost due to potential warranty costs, repairs and loss of market share.

Reliability is the probability that the system will perform its intended function successfully for a specified interval of time, under stated operating and environmental conditions. It is therefore, related to product functionality over time which is determined by the so-called "hard" and "soft" failures [1]. In a hard failure the system loses functionality due to a complete breakdown of one or more of its components, while in a soft failure the system is functional but one or more performance measures are out of conformance. The reliability associated with the hard failure is important for non-repairable systems where the replacement or repair of a failed component is not possible and the failed system is removed from the population. In contrast, repairable systems [2] consist of multiple components which can be repaired or replaced if failed keeping therefore, the original system in the population. In this research, we use time-dependent reliability concepts associated with the so-called first-passage of non-repairable systems. Among its many applications, the approach can be used to reduce the lifecycle cost [3, 4] or to set a schedule for preventive condition-based maintenance [5].

The time-dependent probability of failure (see Eq. 5 for definition), also known as cumulative probability of failure [3, 6], is calculated by the following exact relation using the failure rate

$$P_f(0,T) = 1 - (1 - P_f^i(0)) \exp\left\{ -\int_0^T \lambda(t) \mathrm{d}t \right\}, \quad (1)$$

***Corresponding Author:** 248-370-2686 (O), 248-370-4416 (Fax), mourelat@oakland.edu.

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE	2. REPORT TYPE	3. DATES COVERED
23 JAN 2013	**Journal Article**	**11-10-2013 to 14-12-2013**

4. TITLE AND SUBTITLE
Time-dependent Reliability of Dynamic Systems using Subset Simulation with Splitting over a Series of Correlated Time Intervals

5a. CONTRACT NUMBER
W56HZV-04-2-0001

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
Zhonglai Wang; Zissimos Mourelatos; Jing Li; Amandeep Singh; Igor Baseski

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Mechanical Engineering Department,Oakland University,Rochester,Mi,48309

8. PERFORMING ORGANIZATION REPORT NUMBER
; #23620

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
U.S. Army TARDEC, 6501 E. 11 Mile Rd., Warren, Mi, 48397-5000

10. SPONSOR/MONITOR'S ACRONYM(S)
TARDEC

11. SPONSOR/MONITOR'S REPORT NUMBER(S)
#23620

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES
Proceedings of IDETC/CIE 2013 ASME 2013 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference August 4-7, 2013, Portland, OR, USA

14. ABSTRACT

Time-dependent reliability is the probability that a system will perform its intended function successfully for a specified time. Unless many and often unrealistic assumptions are made, the accuracy and efficiency of time-dependent reliability estimation are major issues which may limit its practicality. Monte Carlo simulation (MCS) is accurate and easy to use but it is computationally prohibitive for high dimensional, long duration, time-dependent (dynamic) systems with a low failure probability. This work addresses systems with random parameters excited by stochastic processes. Their response is calculated by time integrating a set of differential equations at discrete times. The limit state functions are therefore, explicit in time and depend on time-invariant random variables and time-dependent stochastic processes. We present an improved subset simulation with splitting approach by partitioning the original high dimensional random process into a series of correlated, short duration, low dimensional random processes. Subset simulation reduces the computational cost by introducing appropriate intermediate failure sub-domains to express the low failure probability as a product of larger conditional failure probabilities. Splitting is an efficient sampling method to estimate the conditional probabilities. The proposed subset simulation with splitting not only estimates the timedependent probability of failure at a given time but also estimates the cumulative distribution function up to that time with approximately the same cost. A vibration example involving a vehicle on a stochastic road demonstrates the advantages of the proposed approach.

15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Public Release**	**17**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

where $P_f^i(0)$ is the instantaneous probability of failure at the initial time. In the commonly used out-crossing rate approach, the failure rate

$$\lambda(t) = \lim_{dt \to 0} \frac{P(t < T_f \le t + dt \mid T_f > t)}{dt}, \qquad (2)$$

where T_f is the time to failure, is approximated by the up-crossing rate [7]

$$\nu^+(t) = \lim_{\substack{\Delta\tau \to 0, \\ \Delta\tau > 0}} \frac{P\left[g(\mathbf{X}(t),t) - S_t < 0 \cap g(\mathbf{X}(t+\Delta\tau),t+\Delta\tau) - S_t \ge 0\right]}{\Delta\tau}, \qquad (3)$$

(see Equations 5 and 6 for notation) under the assumptions that 1) the probability of having two or more out-crossings in $[t, t+\Delta t]$ is negligible compared to the probability of having exactly one out-crossing, 2) Δt is sufficiently small, and 3) the out-crossings in $[t, t+\Delta t]$ are statistically independent of the previous out-crossings in $[0, t]$. In this case, the number of up-crossings $N^+(t,\alpha)$ for a threshold α is a Poisson process.

The out-crossing rate approach was first introduced by Rice [8] followed by extensive studies [6, 9-13] under the assumption of the out-crossings being statistically independent and Poisson distributed. Hagen and Tvedt [7] suggested a parallel system reliability formulation to compute the out-crossing rate. It uses two successive time-invariant analyses based on FORM, and the binomial cumulative distribution to calculate the probability of the joint event in Eq. (3). This approach was later adopted in the PHI2 method [6]. Methods based on Poisson's distribution and the PHI2 method compute an upper bound of the probability of failure of Eq. (3) [14]. A Monte-Carlo based set theory approach has been also proposed [1, 15] using a similar approach with the PHI2 method. Analytical studies such as in [16, 17, 18] revealed that the PHI2 based approach lacks sufficient accuracy for non-monotonic problems such as vibratory systems. For this reason, analytical approaches were proposed in [16, 19, 20] to accurately estimate the time-dependent probability of failure considering non-monotonic behavior.

Although the out-crossing rate approach can easily estimate the time-dependent probability of failure, it has two potential limitations. First, its accuracy may be poor because of the Poisson process assumption of independent out-crossings and second, it may require a large computational effort. An analytical FORM-based estimation of the up-crossing rate (Eq. 3), with its own accuracy limitations, must be performed at the time instants required by the numerical evaluation of the integral in Eq. (1) (e.g. Gauss-Legendre integration points). If the probability of failure $P_f(0,T)$ is calculated at different times *T*, the computational effort increases because the integration points change. This can increase the computational effort. The first limitation has been recently improved in [18] by considering the correlations between the limit-state function at two time instants. The method estimates the up-crossing rate $\nu^+(t)$ by solving an integral equation involving $\nu^+(t)$ and $\nu^{++}(t,t_1)$, the joint probability of up-crossings in times *t* and t_1 [21].

This paper presents a simulation-based method to estimate the time-dependent probability of failure at different time instances. Monte Carlo simulation (MCS) can handle high-dimensional problems, and general failure definitions allowing us to handle system reliability problems. However, it cannot estimate efficiently small probabilities because the number of samples, and hence the number of system analyses required to achieve a given accuracy, is inversely proportional to the failure probability.

Importance sampling techniques [22] are commonly used to shift the underlying distribution towards the failure region in order to sample rare events more efficiently. They require however, a careful choice of the importance sampling density (ISD), which requires knowledge of the failure region. For low-dimensional uncertain systems with relative simple failure regions, many important sampling methods have been developed (e.g. [22, 23]). However, the application of importance sampling to general dynamic reliability problems where the random excitation is represented by a large number of discrete random variables is still an active research area with limited practicality for such problems.

A MCS approach was proposed in [24] to estimate the time-dependent failure rate over the product lifecycle. The efficiency of the method was further improved using an importance sampling method with a decorrelation length [25] in order to reduce the high dimensionality of the problem.

Subset simulation [26, 27] has been recently developed as an efficient simulation method for computing small failure probabilities for general reliability problems. Its efficiency comes from introducing appropriate intermediate failure sub-domains to express the low probability of failure as a product of larger conditional failure probabilities which are estimated with much less computational effort. In doing so, the probability of a rare event in the original probability space, is replaced by a sequence of probabilities of more frequent events in conditional probability spaces. Because it is very challenging to generate samples in the conditional spaces, subset simulation with Markov Chain Monte Carlo (SS/MCMC) [28, 29] and subset simulation with splitting (SS/S) [30-32] methods have been introduced.

In this paper, we propose an improved subset simulation method with splitting by partitioning the original high dimensional random process into a series of correlated, short duration, low dimensional random processes. The proposed method, called subset simulation with splitting and partitioning in time (SS/SPT), not only estimates the time-dependent probability of failure at a given time (as the original SS/S method does) but also estimates the failure rate and the cumulative distribution function (CDF) up to that time with approximately the same cost. It can also handle dynamic systems with random parameters (e.g. random stiffness of a vehicle suspension system) which the original SS/S approach does not consider. Estimation of the CDF is very important for many practical problems such as preventive maintenance using reliability principles, design for lifecycle cost, etc. Also, metrics such as the failure rate, remaining life, and mean-time-to-failure (MTTF) which are functions of the CDF of time to failure are significant for describing the system reliability and performance characteristics.

The paper is organized as follows. Section 2 defines the dynamic systems we consider and provides a brief introduction of the problem and notation. Section 3 introduces the subset simulation approach and the existing MCMC and splitting sampling schemes. Our proposed SS/STP method is presented in Section 4, including accuracy bounds and computational effort estimation. Section 5 uses an example of a quarter vehicle with uncertain parameters on stochastic terrain to demonstrate the characteristics and advantages of the proposed method. Finally, Section 6 summarizes, concludes and presents future work.

2. TIME DEPENDENT RELIABILITY EVALUATION

We consider the time-dependent reliability of dynamic (rigid-body or vibratory) systems whose equations of motion are usually discretized in time and presented in a state-space form. The discretized equations are time integrated using for example, a Runge-Kutta method or Newmark-beta method [33]. They are expressed as

$$\mathbf{X}(t+\Delta t) = f\left(\mathbf{X}(t), \mathbf{Y}, \mathbf{U}(t), t\right), \qquad (4)$$

where $\mathbf{X}(t) \in \Re^p$ is the vector of uncertain states $x_s(t)$, $s = 1,2,\ldots, p$ at time t, Δt is the integration time step, $\mathbf{Y} \in \Re^q$ is the time-independent vector of random variables (e.g. system parameters $\mathbf{Y}_s \in \Re^{q_s}$ and excitation parameters $\mathbf{Y}_e \in \Re^{q_e}$), and $\mathbf{U}(t) = \mathbf{U}(\mathbf{Y}, t) \in \Re^r$ is the time-dependent vector of excitation random processes (e.g. road elevation at a vehicle tire location through time). Both $\mathbf{X}(t)$ and $\mathbf{U}(t)$ are implicit functions of $\mathbf{Y}$. The trajectories $\mathbf{X} = \{\mathbf{X}(t),\ t \in [0,T]\}$ of all states (sample functions of corresponding random processes) are calculated at discrete time instances $t = t_j$, $j = 0,1,\ldots,N_{\text{step}}$ where N_{step} is the number of time integration steps over the period $[0,T]$. For illustration, if $p = 1$, $q = 2$ and $r = 1$, Eq. (4) becomes $x_1(t+\Delta t)) = f\left(x_1(t), y_1, y_2, u(t), t\right)$ where the single state x_1 at time $t+\Delta t$ is a function of the state at time t, y_1 and y_2 are the realizations of the two random variables Y_1 and Y_2, and $u(t)$ is the value of a sample function of the random process $U(t)$ at time t. Time integration of Eq. (4) provides the system response $x_1 = \{x_1(t),\ t \in [0,T]\}$.

The computational effort to solve for the system states $\mathbf{X}(t+\Delta t)$ at time $t+\Delta t$ as a function of the states $\mathbf{X}(t)$ at time t, is considered one function evaluation. It is important to note that the system states $\mathbf{X}(t+\Delta t)$ are a function of the states $\mathbf{X}(t)$ which are in turn a function of the states $\mathbf{X}(t-\Delta t)$. Thus, in order to calculate $\mathbf{X}(t)$, we need the solution at all previous times. For this reason, counting the number of sample function (or trajectory) evaluations instead of the number of function evaluations is preferable for dynamic systems. This is different from problems with explicit in time limit states where a simple function evaluation can be performed at any time t.

Our goal is to calculate the time-dependent probability of failure

$$P_f(0,T) = P\{\exists t \in [0,T]: g(\mathbf{X}(t), t) \geq S_t\}, \qquad (5)$$

where $g: \Re^p \to \Re$ is a function that maps $\mathbf{X}(t)$ to a response of interest and S_t is a given threshold value. Because of the time-dependent uncertain states in $\mathbf{X}(t)$, $G(t) = g(\mathbf{X}(t), t) - S_t$ is a random process which can be viewed as a collection of random variables at different time instances t. Since we consider a first excursion failure problem in Eq. (5), the failure domain, as well as an event therein, can be defined as

$$F = \left\{ \max_{t \in [0,T]} g(\mathbf{X}(t), t) \geq S_t \right\}. \qquad (6)$$

The system operates properly and is called safe if $g(\mathbf{X}(t), t) < S_t, \forall t \in [0,T]$. The system is considered failed if $\exists t \in [0,T]: g(\mathbf{X}(t), t) \geq S_t$. The equation $g(\mathbf{X}(t), t) - S_t = 0$ is the time-dependent limit state surface (LSS).

We consider a non-repairable system where if $g(\mathbf{X}(t_j), t_j) \geq S_t$ for $t_j \in [0,T]$, the system fails and is removed from the population for $t > t_j$. In this case, Eq. (5) can be approximately rewritten as

$$P_{\mathrm{f}}(0,T) = P\left\{\bigcup_{j=1}^{r} g\left(\mathbf{X}(t_j),t_j\right) \geq S_{\mathrm{t}},\ t_j \in [0,T]\right\} = P\left\{\max_{j=0,1,\ldots,r} g\left(\mathbf{X}(t_j),t_j\right) \geq S_{\mathrm{t}}, t_j \in [0,T]\right\} \quad (7)$$

where r is the number of load random variables generated from the random process $G(t) = g(\mathbf{X}(t),t) - S_{\mathrm{t}}$ at discrete times t_j , $j = 1,2,\ldots,r$. For convenience, if N_{step} is the number of time integration steps, we let $r = N_{\mathrm{step}}$. If the time interval $[0,T]$ is discretized with a uniform time step Δt, we have $N_{\mathrm{step}} = T/\Delta t$. According to Eq. (7), the time-dependent reliability evaluation can be transformed into a time-independent system reliability evaluation involving $r = N_{\mathrm{step}}$ correlated random variables. If Δt is small, the time-independent system reliability problem is of very high dimension since N_{step} is inversely proportional to Δt .

3. SUBSET SIMULATION

This section provides a brief introduction to subset simulation and the existing MCMC and splitting sampling schemes. Given the failure event *F* of Eq. (6), a nested sequence of *m* failure domains is formed using a set of increasing threshold levels $S_{\mathrm{t}}^1 < S_{\mathrm{t}}^1 < \cdots < S_{\mathrm{t}}^m = S_{\mathrm{t}}$. The failure domains F^i for $i = 1,2,\ldots,m$ are defined as

$$F^i = \left\{\max_{t\in[0,T]} g(\mathbf{X}(t),t) \geq S_{\mathrm{t}}^i\right\}, \quad (8)$$

where

$$F^1 \supset F^2 \supset \cdots \supset F^m = F, \quad (9)$$

and

$$F^l = \bigcap_{i=1}^{l} F^i,\ l = 1,2,\ldots,m. \quad (10)$$

Figure 1 shows pictorially the *m* failure domains F^i , $i = 1,2,\ldots,m$ for a hypothetical case with two random variables.

Using the definition of conditional probability, we have [31]

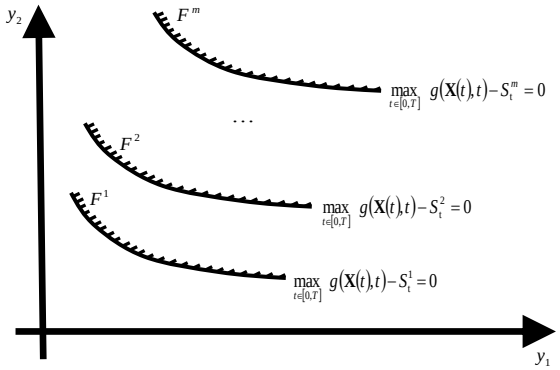


Figure 1. Pictorial representation of failure subdomains

$$\begin{aligned}
P_{\mathrm{f}} &= P(F) = P\left(F^m\right) = P\left(\bigcap_{i=1}^{m} F^i\right) \\
&= P\left(F^m \middle| \bigcap_{i=1}^{m-1} F^i\right) \cdot P\left(\bigcap_{i=1}^{m-1} F^i\right) \\
&= P\left(F^m \middle| F^{m-1}\right) \cdot P\left(\bigcap_{i=1}^{m-1} F^i\right) \\
&= \prod_{i=m-2}^{m-1} P\left(F^{i+1} \middle| F^i\right) \cdot P\left(\bigcap_{i=1}^{m-2} F^i\right) \\
&\cdots \\
&= \prod_{i=2}^{m-1} P\left(F^{i+1} \middle| F^i\right) \cdot P\left(\bigcap_{i=1}^{2} F^i\right) \\
&= P\left(F^1\right) \cdot \prod_{i=1}^{m-1} P\left(F^{i+1} \middle| F^i\right) \\
&= P_{\mathrm{f}}^1 \cdot \prod_{i=2}^{m} P_{\mathrm{f}}^i \\
&= \prod_{i=1}^{m} P_{\mathrm{f}}^i
\end{aligned} \quad (11)$$

where the conditional probability $P\left(F^i \middle| F^{i-1}\right)$, $i = 2,\ldots,m$ is denoted by P_{f}^i and $P\left(F^1\right)$ is denoted by P_{f}^1 . Eq. (11) indicates that the probability of failure P_{f} can be expressed as the product of a sequence of larger conditional probabilities P_{f}^i and P_{f}^1 . Even though P_{f} is very small, the conditional probabilities can be sufficiently large to be evaluated accurately by MCS using a much smaller number of samples for the intermediate failure events F^i , $i = 2,\ldots,m$. For example, if the probability of failure for the rare event F is $P_{\mathrm{f}} = 10^{-6}$, the failure domains F^i , $i = 2,\ldots,6$ can be defined so that the corresponding failure events have probabilities $P_{\mathrm{f}}^1 = 0.1$ and conditional probabilities $P_{\mathrm{f}}^i = 0.1$ for $i = 2,\ldots,6$ such that $P_{\mathrm{f}} = \prod_{i=1}^{6} P_{\mathrm{f}}^i = 10^{-6}$.

The much larger probabilities P^i can be efficiently evaluated using MCS for a given accuracy. The probability calculation of a rare event is therefore, replaced by a sequence of more frequent events in the conditional probability spaces.

To evaluate the conditional probabilities in Eq. (11), a sampling scheme is needed to produce sample functions, which satisfy the conditional events. Two such sampling schemes are Markov Chain Monte Carlo (MCMC) and splitting. Their difference is on how they generate offsprings from appropriate parent samples. A brief introduction to MCMC and splitting is provided next.

3.1 Sampling with MCMC and splitting

Subset simulation requires the conditional probabilities $P^i = P\left(F^i \middle| F^{i-1}\right)$ for $i = 1,2,\ldots,m$ to calculate the overall small probability of failure P_f as the product of the larger conditional probabilities P^i and P^1. Monte Carlo simulation can estimate the conditional probabilities as long as we can generate conditional samples. For example, to calculate $P\left(F^i \middle| F^{i-1}\right)$ we must generate sample functions $g(\mathbf{X}(t),t)$ which satisfy the failure condition $F^{i-1} = \left\{ \max_{t\in[0,T]} g(\mathbf{X}(t),t) \geq S_t^{i-1} \right\}$. For that, Markov Chain Monte Carlo (MCMC) or splitting (S) sampling schemes can be used.

To generate MCMC sample functions, we perturb a "parent" sample function $g_P(\mathbf{X}(t),t)$ which satisfies the condition of the F^{i-1} domain (thick line in Figure 2) to generate an offspring $g_O(\mathbf{X}(t),t)$ (thin lines in Figure 2). However, some of the offsprings may not belong to the F^{i-1} domain (thin solid line in Figure 2) and are therefore, rejected. As a result, the MCMC sampling process can be inefficient. More details are provided in [26, 30, 31].

Generating offsprings using splitting does not have the drawback of MCMC. Figure 3(a) highlights the main idea. Considering that the "parent" sample function $g_P(\mathbf{X}(t),t)$ belongs to the F^{i-1} domain, we can partition it into two partial signals $g_P^- = [g_P(\mathbf{X}(t),t), t \leq t_{pass^1}]$ and $g_P^+ = [g_P(\mathbf{X}(t),t), t > t_{pass^1}]$, where t_{pass^1} is the time when $g_P(\mathbf{X}(t),t)$ crosses the threshold S_t^{i-1} and $t_{pass^{1-}} = t_{pass^1} - \Delta t$ is the time instant right before the first passage time t_{pass^1} (see Figure 3a), and record the system states $\mathbf{S}_{pass^{1-}} = \mathbf{X}\left(t_{pass^1} - \Delta t\right)$. We assume 1) that the sample functions of the excitation random processes $\mathbf{U}(t)$ can be partitioned similarly into $\mathbf{U}_P^- = [\mathbf{U}(t), t \leq t_{pass^1}]$ and $\mathbf{U}_P^+ = [\mathbf{U}(t), t > t_{pass^1}]$ and 2) that the new sample functions $\mathbf{U}_O^+ = [\mathbf{U}(t), t > t_{pass^1}]$ of the excitation can be generated after t_{pass^1}. Then, using the states $\mathbf{S}_{pass^{1-}}$ and the excitation $\mathbf{U}_O^+$, we generate an offspring $\left(g_P^-, g_O^+\right)$, where

$$g_O^+ = [g\left(\mathbf{X}(t), \mathbf{U}_O^+(t), t\right), t \geq t_{pass^1}].$$

This process guarantees that the offspring $\left(g_P^-, g_O^+\right)$ belongs to the F^{i-1} domain.

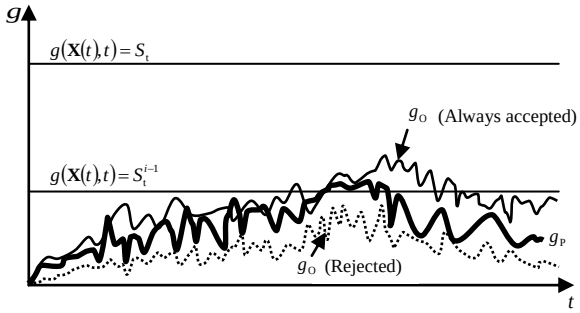
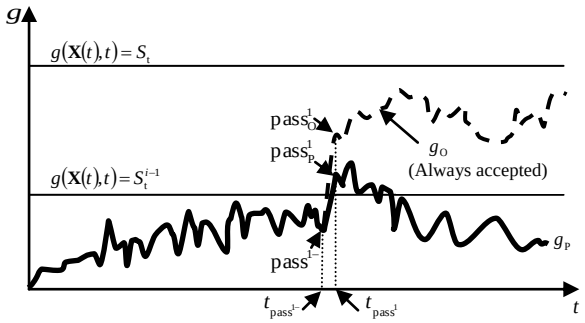
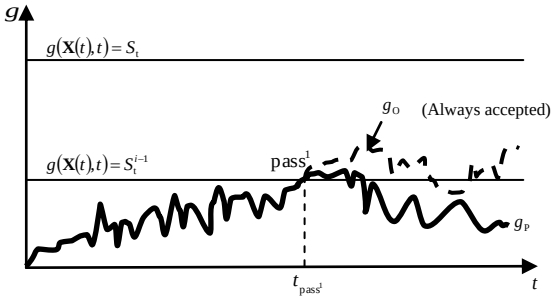


Figure 2. Accepted and rejected sample functions generated by MCMC



(a)



(b)

Figure 3. Sample functions generated by splitting

If the time step Δt is small enough, the difference among $\text{pass}_{\text{O}}^{1}$, $\text{pass}_{\text{P}}^{1}$, and pass^{1-} is very small and the splitting point pass^{1-} (Figure 3a) can be replaced with $\text{pass}_{\text{P}}^{1} = \text{pass}^{1}$ (Figures 3a and 3b). Then, the generated offspring is always accepted without a path dependency issue. However if the time step Δt is not small enough, $\text{pass}_{\text{O}}^{1}$, $\text{pass}_{\text{P}}^{1}$, and/or pass^{1-} may be quite off from the threshold S_{t}^{i-1} and the offspring will have a path dependency issue if the splitting starts at $\text{pass}_{\text{P}}^{1}$. To avoid path dependency and improve accuracy, we perform splitting at pass^{1-} similarly to [32]. This may compromise efficiency because of a repeated generation of response realizations until the generated offspring path has a first passage point in the time period of interest.

4. PROPOSED APPROACH

Subset simulation with splitting (SS/S) or subset simulation with MCMC (SS/MCMC), are promising approaches in estimating the reliability of dynamic systems with low probability of failure. Even though SS/S is more efficient than SS/MCMC, its computational efficiency decreases rapidly when several probabilities of failure (e.g., $P_f(0,T_1)$, $P_f(0,T_2)$ and $P_f(0,T_3)$ with $T_1 < T_2 < T_3$) are needed to build the CDF, because each probability of failure is estimated independently starting from $t = 0$.

To address this issue for non-repairable systems, we propose a subset simulation method with splitting and partitioned time (SS/SPT). The method partitions $[0,T]$ into several time intervals and uses a modified SS/S approach sequentially using the survived (not failed) sample trajectories from the previous time interval to directly obtain the failure rate in the current time interval, and then calculate the cumulative probability of failure. A nested sequence of *m* failure regions $F^1 \supset F^2 \supset \cdots \supset F^m = F$ is formed for different threshold levels $S_{\text{t}}^1 < S_{\text{t}}^2 < \cdots < S_{\text{t}}^m = S_{\text{t}}$. The random processes within different time intervals are correlated because if a failure occurs outside a time interval, the failed system is disposed at that time.

The time of interest $[0,T]$ is discretized at instances $\{0 \;\; T_1 \;\; T_2 \;\; \cdots \;\; T_n \;\; \cdots \;\; T_{N_{\text{interval}}} = T\}$ so that $T = \sum_{n=1}^{N_{\text{interval}}} (T_n - T_{n-1}) = N_{\text{interval}} \Delta T$ with $T_0 = 0$, $T_{N_{interval}} = T$ and $\Delta T = T / N_{\text{interval}}$. A series of N_{interval} correlated/conditional failure regions are then formed defined by the following events

$$F_{\text{c}n} = \left\{ \max_{t \in [T_{n-1}, T_n]} g(\mathbf{X}(t), t) \geq S_{\text{t}} \;\middle|\; \overline{F}_{n-1} \right\}, \qquad (12a)$$

where

$$\overline{F}_{n-1} = \left\{ \max_{t \in [0, T_{n-1}]} g(\mathbf{X}(t), t) < S_{\text{t}} \right\}, \qquad (12b)$$

and

$$F_n = \left\{ \max_{t \in [0, T_n]} g(\mathbf{X}(t), t) \geq S_{\text{t}} \right\} \qquad (12c)$$

for $n = 1,2,\ldots,N_{\text{interval}}$.

The dependencies in the conditional failure events $F_{\text{c}n}$ in the time direction are simply enforced during simulation by using only the surviving (not failed) sample functions among all sample functions in $[0,T_{n-1}]$ in the subsequent time period $[T_{n-1},T_n]$. This allows us to partition time T in shorter time intervals $[T_{n-1},T_n], n = 1,2,\ldots,N_{\text{interval}}$ and use the subset simulation with splitting in each of the time intervals to calculate the conditional probabilities

$$P_{\text{cf}n} = P(F_{\text{c}n}). \qquad (13)$$

For the time interval $[T_{n-1},T_n]$, we use $F_{\text{c}n}^i$ instead of F^i to represent the i^{th} subset event in the threshold direction

$$F_{\text{c}n}^i = \left\{ \max_{t \in [T_{n-1}, T_n]} g(\mathbf{X}(t), t) \geq S_{\text{t}}^i \;\middle|\; \overline{F}_{n-1} \right\}, \qquad (14)$$

where $i = 1,2,\ldots,m_n$, and m_n is the number of intermediate thresholds for the n^{th} time interval $[T_{n-1},T_n]$. The corresponding probability is denoted by $P_{\text{cf}n}^i = P(F_{\text{c}n}^i)$. The probability $P_{\text{cf}n} = P(F_{\text{c}n})$ is calculated using Eq. (11), where $F_{\text{c}n} = \bigcap_{i=1}^{m_n} F_{\text{c}n}^i$.

The probability $P_{\text{cf}n}$ can be also used to calculate the failure rate

$$\lambda_n = P_{\text{cf}n} / \Delta T_n \qquad (15)$$

in the time period $[T_{n-1},T_n]$ where $\Delta T_n = T_n - T_{n-1}$, because it represents the probability of failure in the time period $[T_{n-1},T_n]$ under the condition that the system is safe in $[0,T_{n-1}]$.

Because $F_n = F_{n-1} \cup (\overline{F}_{n-1} F_{\text{c}n})$, we can use the modified SS/S approach to calculate $P_{\text{cf}n}$ in $[T_{n-1},T_n]$ and the probabilities of failure

$$P_{\text{f}n} = P(F_n) = P_{\text{f}n-1} + (1 - P_{\text{f}n}) P_{\text{cf}n} \qquad (16)$$

at all instances T_n , $n=1,2,\ldots,T_{\text{interval}}$ with almost the same computational effort the SS/S approach requires to calculate the probability of failure at the final time *T*. Details on the computational effort are provided in Section 4.1.

Figure 4 highlights the basic idea of the SS/SPT method. At most *N* trajectories are simulated in $[0,T]$. The simulation is first carried out in $[0,T_1]$. For illustration purposes, Figure 4 shows only two trajectories. Because both trajectories cross the intermediate threshold S_{t}^1, they are used to generate offsprings (dotted lines) in $[0,T_1]$. The offsprings along with the parent trajectories, are used to calculate the probability of crossing the next intermediate threshold S_{t}^2. The system states $\mathbf{X}(t)$ at points A and B are recorded in order to continue if necessary, the simulation of the particular trajectory for the next time interval $[T_1,T_2]$. The process is repeated for all intermediate thresholds in $[0,T_1]$. All offsprings are generated only within the current time interval.

Because trajectory 1 crosses the overall threshold S_{t} in $[0,T_1]$, it is considered failed and removed from the population. Thus, we do not continue its simulation in $[T_1,T_2]$. In contrast, because trajectory 2 did not cross S_{t} in $[0,T_1]$, we continue its simulation in $[T_1,T_2]$ using the states $\mathbf{X}(t)$ at point B. In $[T_1,T_2]$, the second trajectory crosses both S_{t}^1 and S_{t}^2, and can produce offsprings as shown in Figure 4. The process of continuing the simulation of only the survived parent trajectories, which can subsequently create offsprings, is carried out for all time intervals.

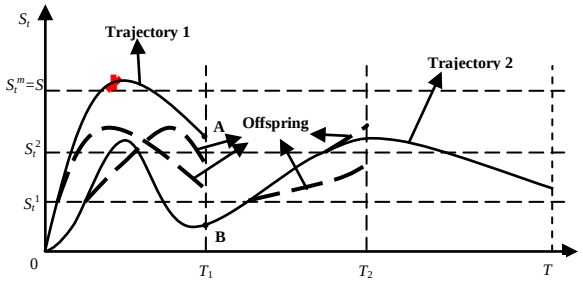


Figure 4. Schematic of basic features of the SS/SPT method

A step-by-step description of the proposed SS/SPT algorithm is given below followed by the developed SS/SPT algorithm.

Step 1. *Initialize algorithmic parameters.*

Step 2. *Generate* $N=N_{\text{sn}}$ *replications of the system random parameters.*

For each time interval, Steps 3 through 8 are performed:

Step 3. *Generate a state trajectory for each system parameter replication for a random excitation trajectory.*

Step 4. *Determine the first threshold level for failure event* F^1 *(see Equation 12c), and calculate the corresponding probability of failure* $P_{\text{f}\,n}^1=P\left(F^1\right)$.

Step 5. *Use splitting to generate offspring trajectories from randomly selected "parent" trajectories for failure event* F^i .

Step 6. *Determine the first threshold level of failure event* F^{1+i} *and calculate the corresponding probability of failure* $P_{\text{f}\,n}^{1+i}=P\left(F^{1+i}\right)$.

Step 7. *Calculate the failure rate and cumulative probability of failure.*

Step 8. *Reset for next time interval.*

A pseudo code description of the proposed SS/SPT algorithm is provided below with details for all the above steps.

Step 1: *Initialize algorithmic parameters.*

Specify the guessed probability of failure P_{fn}^{guess} (P_{fn} in Equation 16).

Specify the guessed expected conditional probability $\bar{p}_n^{guess}$ for each subset simulation level.

Determine the expected conditional probability $\bar{p}$ for each subset simulation level

$$\bar{p}=\max\left\{\bar{p}_n^{guess},P_{fn}^{guess}\right\}.$$

Specify the number of the initial independent parent response trajectories (sample functions) N_{sn} .

$N=N_{\text{sn}}$;

Specify the life span of interest T .

Set $T_0=0$.

Specify a uniform time interval T_{interval} to partition T .

Calculate $N_{\text{interval}}=\text{ceil}\left(T/T_{\text{interval}}\right)$ and reset $T=N_{\text{interval}}T_{\text{interval}}$.

Specify the initial values of excitation $\mathbf{U}_0^{(k)}$ for $k=1,2,\ldots,N_{\text{sn}}$.

Specify the initial values of the states $\mathbf{X}_0^{(k)}$ for $k=1,2,\ldots,N_{\text{sn}}$.

Set $P_{\text{f}\,0}=0$ by assuming that the system has a very small failure probability at the

initial time $T_0 = 0$.

Step 2: *Generate $N = N_{sn}$ replications of the system random parameters.*

For $k = 1:N$

Generate independent random system parameters $\mathbf{Y}_s^{(k)}$.

End

For $n = 1:N_{\text{interval}}$

Step 3: *Generate one state trajectory for each system parameter replication for a random excitation trajectory.*

For $k = 1:N$

For time period $[T_{n-1}, T_n]$ generate independent random excitation parameters $\mathbf{Y}_e^{(k)}$ and the corresponding excitation trajectory $\mathbf{U}^{(k)}(t)$; set $\mathbf{Y}^{(k)} = \left[\mathbf{Y}_s^{(k)}; \mathbf{Y}_e^{(k)}\right]$; solve for state trajectory (sample function) $\mathbf{X}^{(k)}(t)$ using Equation (4) with initial conditions $\mathbf{U}_0^{(k)}$ and $\mathbf{X}_0^{(k)}$. Calculate the corresponding response $g\left(\mathbf{X}^{(k)}(t), t\right)$.

End

Let N_s be the number of trajectories that satisfy $\left\{ \max_{t\in[T_{n-1},T_n]} g^{(k)}(\mathbf{X}(t),t) < S_t,\ k = 1,2,\ldots,N \right\}$.

Denote and record the corresponding excitation, state, and system parameter vectors by $\mathbf{U}_s^{(k_s)}(t)$, $\mathbf{X}_s^{(k_s)}(t)$ and $\mathbf{Y}_{ss}^{(k_s)}$ respectively, for $k_s = 1,2,\ldots,N_s$.

Step 4: *Determine the first threshold level for failure event F^1 (see Eq. 12c), and calculate the corresponding probability of failure $P_{f\,n}^1 = P(F^1)$.*

$i = 0$;

Let S_t^{1+i} be the $(1-\bar{p})\times 100$ percentile of

$$\left\{ \max_{t\in[T_{n-1},T_n]} g^{(k)}(\mathbf{X}(t),t),\ k = 1,2,\ldots,N \right\}.$$

Record the time coordinates and state values of the trajectories (sample functions) at the first-passage points over the intermediate threshold $g\left(\mathbf{X}^{(k)}(t),t\right) = S_t^{1+i}$ and record the corresponding system parameters. Instead of using the approximation $N_f \approx \bar{p}N$, calculate N_f by counting the exact number of all trajectories satisfying $g\left(\mathbf{X}^{(k)}(t),t\right) \geq S_t^{1+i}$. Denote the time coordinates, state values, and the corresponding system parameters right before the first passage times by $\left\{ \left[t_{\text{pass}^{1-}}^{(k_f)}, \mathbf{S}_{\text{pass}^{1-}}^{(k_f)}, \mathbf{Y}_{s\,\text{pass}^{1-}}^{(k_f)} \right],\ k_f = 1,2,\ldots,N_f \right\}$ and record the trajectories $\left\{ \mathbf{X}^{(k_f)},\ k_f = 1,2,\ldots,N_f \right\}$ reaching the threshold.

Calculate $P_{f\,n}^{1+i} = N_f / N$.

While $S_t^{1+i} < S_t$

$i = i + 1$;

Step 5: *Use splitting to generate offspring trajectories from randomly selected "parent" trajectories for failure event F^i.*

For $k = (N_f + 1):N$

Randomly select values for the tuple $\left[\bar{t}, \bar{\mathbf{S}}, \bar{\mathbf{Y}}_s\right]$ from $\left\{ \left[t_{\text{pass}^1}^{(k_f)}, \mathbf{S}_{\text{pass}^1}^{(k_f)}, \mathbf{Y}_{s\,\text{pass}^1}^{(k_f)} \right],\ k_f = 1,2,\ldots,N_f \right\}$ with a uniform probability of $1/N_f$.

Generate $\left\{ \mathbf{X}^{(k)}(t),\ t = \left[\bar{t}, \bar{t} + \Delta t\right] \right\}$ starting from $\bar{\mathbf{S}}$ using $\bar{\mathbf{Y}}_s$ and the newly generated excitation $\mathbf{U}(t)$ according to Equation (4). If $g\left(\mathbf{X}(\bar{t} + \Delta t), \bar{t} + \Delta t\right) < S_t^{1+i}$, redo this step for the same k until $g\left(\mathbf{X}(\bar{t} + \Delta t), \bar{t} + \Delta t\right) \geq S_t^{1+i}$. For the first passage, denote the time coordinate, state value, and the corresponding system parameters by $\left[t_{\text{pass}^1}^{(k)}, \mathbf{S}_{\text{pass}^1}^{(k)}, \mathbf{Y}_{s\,\text{pass}^1}^{(k)} \right]$ (noting that $t_{\text{pass}^1}^{(k)} = \bar{t} + \Delta t$ and $\mathbf{Y}_{\text{pass}^1}^{(k)} = \mathbf{Y}_s$). Record this one time step trajectory as $\mathbf{X}^{(k)}$.

Continue the trajectory $\mathbf{X}^{(k)}$ by generating $\left\{ \mathbf{X}^{(k)}(t),\ t = \left[\bar{t}_{\text{pass}^1}^{(k)}, T_n\right] \right\}$ starting from $\mathbf{S}_{\text{pass}^1}^{(k)}$ using $\mathbf{Y}_{\text{pass}^1}^{(k)}$ and the newly generated excitation $\mathbf{U}^{(k)}(t)$ according to Equation (4).

End

Step 6: *Determine the first threshold level of failure event F^{1+i} and calculate the corresponding probability of failure $P_{f\,n}^{1+i} = P(F^{1+i})$.*

Let S_t^{1+i} be the $(1-\bar{p})\times 100$ percentile of

$\left\{ \max_{t\in[T_{n-1},T_n]} g^{(k)}\left(\mathbf{X}(t),t\right), k=1,2,\ldots,N \right\}$.

Record the system parameters, time coordinates and state values of the trajectories exactly at the first-passage over the intermediate threshold $g\left(\mathbf{X}^{(k)}(t),t\right)=S_{\mathrm{t}}^{1+i}$. Instead of using the approximation $N_{\mathrm{f}}\approx \bar{p}N$, calculate N_{f} by counting the exact number of all trajectories satisfying $g\left(\mathbf{X}^{(k)}(t),t\right)\geq S_{\mathrm{t}}^{1+i}$. Denote the first passage time coordinates, state values and the corresponding system parameters by $\left\{ \left[t_{\mathrm{pass}^1}^{(k_{\mathrm{f}})}, \mathbf{S}_{\mathrm{pass}^1}^{(k_{\mathrm{f}})}, \mathbf{Y}_{\mathrm{spass}^1}^{(k_{\mathrm{f}})} \right], k_{\mathrm{f}}=1,2,\ldots,N_{\mathrm{f}} \right\}$ and record the trajectories $\left\{ \mathbf{X}^{(k_{\mathrm{f}})}, k_{\mathrm{f}}=1,2,\ldots,N_{\mathrm{f}} \right\}$ reaching the threshold.

Calculate $P_{\mathrm{f}\,n}^{1+i}=N_{\mathrm{f}}/N$

End

$i=i+1$

Step 7: *Calculate the failure rate and cumulative probability of failure.*

Let N_{f} be the number of trajectories that satisfy

$\left\{ \max_{t\in[T_{n-1},T_n]} g^{(k)}\left(\mathbf{X}(t),t\right)\geq S_{\mathrm{t}}, k=1,2,\ldots,N \right\}$.

$P_{\mathrm{f}\,n}^{m}=N_{\mathrm{f}}/N$;

$P_{\mathrm{cf}\,n}=\prod_{i=1}^{m} P_{\mathrm{f}\,n}^{i}$.

$\lambda_n=P_{\mathrm{cf}\,n}/T_{\mathrm{interval}}$;

$P_{\mathrm{f}\,n}=P_{\mathrm{f}\,n-1}+\left(1-P_{\mathrm{f}\,n}\right)P_{\mathrm{cf}\,n}$;

Step 8: *Reset for next time interval.*

$N=N_{\mathrm{s}}$;

$\mathbf{U}_0^{(k)}=\mathbf{U}_{\mathrm{s}}^{k}(T_n)$, $\mathbf{X}_0^{(k)}=\mathbf{X}_{\mathrm{s}}^{(k)}(T_n)$ and $\mathbf{Y}_{\mathrm{s}}^{(k)}=\mathbf{Y}_{\mathrm{ss}}^{(k)}$ for $k=1,2,\ldots,N$;

Revise the expected conditional probability $\bar{p}$ for each subset simulation level

$\bar{p}=\max\left\{\bar{p}_n^{guess}, P_{fn}^{guess}\right\}$

End

4.1 Computational Effort

At the n^{th} time interval, we simulate N_{sn} parent trajectories within $[T_{n-1},T_n]$ by solving the dynamic Equations (4). Using a constant time integration step Δt , the number of function evaluations (see paragraph above Equation 5 for definition) per trajectory within $[T_{n-1},T_n]$ is $\frac{T_n-T_{n-1}}{\Delta t}=\frac{T}{N_{\mathrm{interval}}\Delta t}$, where N_{interval} is the number of time intervals we partition the total time $[0,T]$ into. This results in an upper limit $N_{sn}\frac{T}{N_{\mathrm{interval}}\Delta t}$ of function evaluations for the parent trajectories.

At the i^{th} threshold of the n^{th} time interval, we simulate

$$\left(N_{sn}-N_{\mathrm{f}\,n}^{i}\right)=\left(N_{sn}-\bar{p}_n N_{sn}\right)=\left(1-\bar{p}_n\right)N_{sn} \quad (17)$$

offspring trajectories in $[T_{n-1},T_n]$ from $\bar{t}$ to T_n by solving the dynamic Equations (4) where $\bar{t}\in[T_{n-1},T_n]$ is the time a trajectory crosses the i^{th} threshold. In Equation (17), $N_{\mathrm{f}\,n}^{i}$ is the number of the sample functions among N_{sn} which cross the i^{th} threshold and $\bar{p}_n$ is the desired conditional probability for each subset level. Based on the definition of $\bar{t}$, it is reasonable to assume $\bar{t}\approx\frac{T_{n-1}+T_n}{2}$. The upper limit of function evaluations per offspring trajectory at the i^{th} threshold of the n^{th} time interval is therefore, equal to $\frac{T_n-\bar{t}}{\Delta t}\approx\frac{T_n-T_{n-1}}{2\Delta t}=\frac{T}{2N_{\mathrm{interval}}\Delta t}$. If m_n is the number of thresholds where offsprings are generated, we need up to $\left(1-\bar{p}_n\right)N_{sn}\frac{T}{2N_{\mathrm{interval}}\Delta t}\left(m_n-1\right)$ function evaluations for all $\left(m_n-1\right)$ thresholds.

The upper limit of the total number of function evaluations for the n^{th} time interval $[T_{n-1},T_n]$ is thus equal to $N_{sn}\frac{T}{N_{\mathrm{interval}}\Delta t}+\left(1-\bar{p}_n\right)N_{sn}\frac{T}{2N_{\mathrm{interval}}\Delta t}\left(m_n-1\right)$ or

$$\left(1+\frac{\left(1-\bar{p}_n\right)\left(m_n-1\right)}{2}\right)\frac{T}{N_{\mathrm{interval}}\Delta t}N_{sn}. \quad (18)$$

The number of function evaluations in relation (18) is approximately the same for all N_{interval} time intervals, yielding the following upper bound for the overall number of function evaluations

$$N_{\text{interval}}\left(1+\frac{(1-\bar{p}_n)(m_n-1)}{2}\right)\frac{T}{N_{\text{interval}}\Delta t}N_{sn} = \left(1+\frac{(1-\bar{p}_n)(m_n-1)}{2}\right)\frac{T}{\Delta t}N_{sn} \quad (19)$$

Considering that $T/\Delta t$ represents the number of function evaluations to generate a single trajectory (sample function) of the output random process from 0 to *T*, the upper limit of an equivalent number of simulated sample functions is

$$\left(1+\frac{(1-\bar{p}_n)(m_n-1)}{2}\right)N_{sn}. \quad (20)$$

Relation (18) provides the approximate number of function evaluations for the subset simulation with splitting (SS/S) method if $N_{\text{interval}}=1$. This results in the same computational effort with the proposed SS/SPT approach (Equation 20) in terms of equivalent number of simulated sample functions. However, the SS/SPT method calculates the entire CDF and the failure rate as a function of time while the SS/S method simply provides the probability of failure at the final time *T*. This can be a significant advantage of the proposed SS/SPT over the direct SS/S.

4.2 Estimation of Number of Parent Trajectories and Intermediate Thresholds

This section estimates the bounds of the coefficient of variation (C.O.V.) $\delta_{P_{\text{cf}n}}$ of the probability estimate $\hat{P}_{\text{cf}n}$ of $P_{\text{cf}n}$ (see Equations 13 and 12) for the time interval $[T_{n-1},T_n], n=1,2,\ldots,N_{\text{interval}}$ where $T_0=0$. The bounds are used to estimate the appropriate number of parent trajectories N_{sn}, the expected conditional probability of failure $\bar{p}_n$ for each subset level, and the number of intermediate thresholds m_n for each time interval for a given C.O.V.

In subset simulation, the C.O.V. $\delta_{P_{\text{cf}n}}$ of $P_{\text{cf}n}$ is approximated by

$$(\delta_{P_{\text{cf}n}})^2 \approx \sum_{i=1}^{m_n}(\delta_{P^i_{\text{cf}n}})^2, \quad (21)$$

where $\delta_{P^i_{\text{cf}n}}$ is the C.O.V. of the probability estimate $\hat{P}^i_{\text{cf}n}$ at the i^{th} threshold. According to [31], $\delta_{P^i_{\text{cf}n}}$ is given as

$$\delta_{P^i_{\text{cf}n}} = \sqrt{\frac{1-P^i_{\text{cf}n}}{P^i_{\text{cf}n}N^i_n}(1+\gamma^i_n)}. \quad (22)$$

In our proposed SS/SPT approach, $\gamma^1_n=0$ and γ^i_n is a constant provided by

$$0\le\gamma^i_n\le E\left(\frac{N^i_n}{N^i_n-N^i_{\text{f}n}}\right) \quad (23)$$

for $i=2,\ldots,m_n$, where N^i_n is the total number of sample functions for failure domain F^{i-1} and $N^i_{\text{f}n}$ is the number of sample functions among N^i_n which are also used in the failure domain F^i. For simplicity, we assume that each subset level has the same expected conditional probability $P^i_{\text{cf}n}=\bar{p}_n$. We also assume that the total number of sample functions N^i_n is the same for each subset level *i* and equal to N_{sn} (i.e., $N^i_n=N_{sn}$) which must be estimated. This assumption results in the same number of conditional failure samples $N^i_{\text{f}n}=\bar{p}_nN_{sn}$ for each subset level. Therefore, the expected number of offspring trajectories, generated from the $N^i_{\text{f}n}$ sample functions, and reaching the i^{th} threshold is

$$N^i_{on}=N_{sn}-N^i_{\text{f}n}=(1-\bar{p}_n)N_{sn}, \quad (24)$$

and Equation (23) becomes

$$0\le\gamma^i_n\le E\left(\frac{1}{1-\bar{p}_n}\right)=\frac{1}{1-\bar{p}_n} \quad (25)$$

for $i=2,\ldots,m_n$

Combining Equations (21), (22) and (25) and using the assumption $P^i_{\text{cf}n}=\bar{p}_n$, the C.O.V. $\delta_{P_{\text{cf}n}}$ of the probability estimate $\hat{P}_{\text{cf}n}$ for the n^{th} time interval, has the following bounds

$$\sum_{i=1}^{m_n}(\bar{\delta}_n)^2\le(\delta_{P_{\text{cf}n}})^2\le(\bar{\delta}_n)^2+\sum_{i=2}^{m_n}(\bar{\delta}_n)^2\left(1+\frac{1}{1-\bar{p}_n}\right), \quad (26)$$

where

$$\bar{\delta}_n=\sqrt{\frac{1-\bar{p}_n}{\bar{p}_nN_{sn}}}. \quad (27)$$

Substitution of Equation (27) in Equation (26) yields

$$m_n\frac{1-\bar{p}_n}{\bar{p}_nN_{sn}}\le(\delta_{P_{\text{cf}n}})^2\le\left[1+(m_n-1)\frac{2-\bar{p}_n}{1-\bar{p}_n}\right]\frac{1-\bar{p}_n}{\bar{p}_nN_{sn}}. \quad (28)$$

Based on Equation (11) and the assumption $P^i_{\text{cf}n}=\bar{p}_n$, the number of threshold levels m_n for the n^{th} time interval in Eq. (28) can be calculated from

$$P_{\text{cf}n}=\prod_{i=1}^{m_n}P^i_{\text{cf}n}=\prod_{i=1}^{m_n}\bar{p}_n=(\bar{p}_n)^{m_n}$$

or

$$\ln(P_{\text{cf}n})=m_n\ln(\bar{p}_n)\Rightarrow m_n=\text{ceil}\left(\frac{\ln P_{\text{cf}n}}{\ln\bar{p}_n}\right). \quad (29)$$

Equations (28) and (29) are used to estimate the bounds of $\delta_{P_{\text{cf}n}}$ based on estimates of N_{sn} and $P_{\text{cf}n}$, and

a predetermined value of $\bar{p}_n$. Alternatively for a given $\delta_{P_{cfn}}$, we can estimate the bounds of N_{sn}.

As an example, we assume that the time internal of interest is $[0,T]=[0,352\text{sec}]$ and the estimated probability of failure $P_{fn}=P(F_n)$ (see Eq. 12c) at T = 352 sec is $P_{fn}(0,352)=0.2$. If we partition the interval $[0,T]=[0,352\text{sec}]$ at every 8 seconds (i.e., $N_{\text{interval}}=\frac{352}{8}=44$ time intervals), P_{cfn} is estimated by $P_{cfn}=\frac{P_{fn}}{N_{\text{interval}}}=\frac{0.2}{44}=0.004545$. If we also assume $\bar{p}_n=0.2$, Equation (29) estimates the number of subset levels $m_n=\text{ceil}\left(\frac{\ln P_{cfn}}{\ln \bar{p}_n}\right)=\text{ceil}\left(\frac{\ln(0.004545)}{\ln(0.2)}\right)=4$ for each time interval. Then Equation (28) can provide the bounds of N_{sn} for an assumed C.O.V. of $\delta_{P_{cfn}}=0.07$ as

$$m_n\frac{1-\bar{p}_n}{\bar{p}_n(\delta_{P_{cfn}})^2}\le N_{sn}\le\left[1+(m_n-1)\frac{2-\bar{p}_n}{1-\bar{p}_n}\right]\frac{1-\bar{p}_n}{\bar{p}_n(\delta_{P_{cfn}})^2}$$ or

$$4\frac{1-0.2}{0.2\cdot 0.07^2}\le N_{sn}\le\left[1+(4-1)\frac{2-0.2}{1-0.2}\right]\frac{1-0.2}{0.2\cdot 0.07^2}\Rightarrow$$

$3265\le N_{sn}\le 6326$. Choosing $N_{sn}=5000$, the relation (20) gives the approximate number of $\left(1+\frac{(1-\bar{p}_n)(m_n-1)}{2}\right)N_{sn}=\left(1+\frac{(1-0.2)\cdot(4-1)}{2}\right)\cdot 5000$ $=11{,}000$ simulated trajectories.

5. VEHICLE VIBRATION EXAMPLE

The linear representation of a quarter vehicle in Fig. 5 is employed to illustrate the proposed SS/SPT method. The vehicle travels over a stochastic terrain at 20 miles per hour (mph). The system has two random parameters; the damping coefficient b_s and the stiffness coefficient k_s. Both are normally distributed with $b_s\sim N(7000,1400^2)$N/m/s and $k_s\sim N(40000,4000^2)$ N/m. The sprung and unsprung masses m_s and m_u of the suspension, the tire stiffness coefficient k_t and the tire damping coefficient b_t are deterministic with $m_s=100$kg, $m_u=100$kg, $k_t=40\times10^4$N/m and $k_t=40\times10^4$N/m, $b_t=40\times10^3$ N/m/s. The vehicle is travelling over a stochastic terrain which provides excitation. The elevation $u(t)$ of the terrain is a random process.

The state-space approach is used to determine the vertical acceleration $g(\mathbf{X}(t),t)=\ddot{x}_s(t)$ of the sprung mass, in g's. Failure occurs if the magnitude of the vertical acceleration exceeds a threshold S_t (i.e., $|g(\mathbf{X}(t),t)|\ge S_t$).

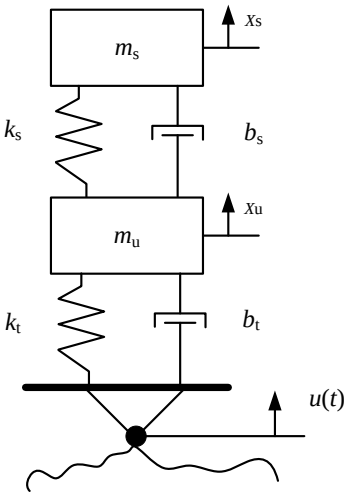


Figure 5. Quarter vehicle model

A third order, autoregressive time series model AR(3) is used to characterize the road excitation process [25]. Time series modeling can characterize both stationary and non-stationary processes [34, 35]. The road model for this example is expressed as

$u_j=1.2456u_{j-1}-0.2976u_{j-2}-0.1954u_{j-3}+\varepsilon_j(0,0.5132^2)$

where the subscript j indicates the time step of the discretized time, $\varepsilon_j(0,05132^2)$ is Gaussian white noise with a standard deviation of 0.5132. The coefficients 1.2456, -0.2976, and -0.1954 are the three estimated feedback parameters. of the AR(3) model.

The following two differential equations comprise the equations of motion

$$m_u\ddot{x}_u+(b_t+b_s)\dot{x}_u-b_s\dot{x}_s+(k_t+k_s)x_u-k_sx_s=k_tu+b_t\dot{u}$$
$$m_s\ddot{x}_s+b_s(\dot{x}_s-\dot{x}_u)+k_s(x_s-x_u)=0$$

which are transformed to state-space form and integrated in time to obtain the response ($x_s(t)$ and $x_u(t)$) which is then used to determine the vertical acceleration $\ddot{x}_s(t)$ of the sprung mass.

The time step for the numerical integration of the above equations is equal to $\Delta t=0.01$ seconds. The threshold for failure is $S_t=3.5$ g's. We calculated the failure rate and the CDF in the time period $[0,352]$ seconds. The latter was partitioned every 2 seconds ($N_{\text{interval}}=176$), 4 seconds ($N_{\text{interval}}=88$) and 8 seconds ($N_{\text{interval}}=44$). For the 2 second partitioning for example, the probability of failure is calculated at 2, 4, 8, … , 352 seconds.

Following the process in the last paragraph of Section 4.2, we can estimate the number of parent

trajectories N_{sn} , the number of intermediate thresholds (or subset levels) m_n and an approximate number of simulated trajectories for a desired C.O.V. of $\delta_{P_{cfn}} = 0.07$ and expected conditional probability $\bar{p}_n = 0.2$. If the time [0, 352] seconds is partitioned with an 8 sec interval, we obtain $4081 \le N_{sn} \le 8163$, $m_n = 5$ and an approximate number of 15,000 simulated trajectories. For partitioning with 4 or 2 seconds, we get $3265 \le N_{sn} \le 6326$, $m_n = 4$ and an approximate number of 11,000 simulated trajectories. Based on these estimates, we selected $N_{sn} = 5{,}000$.

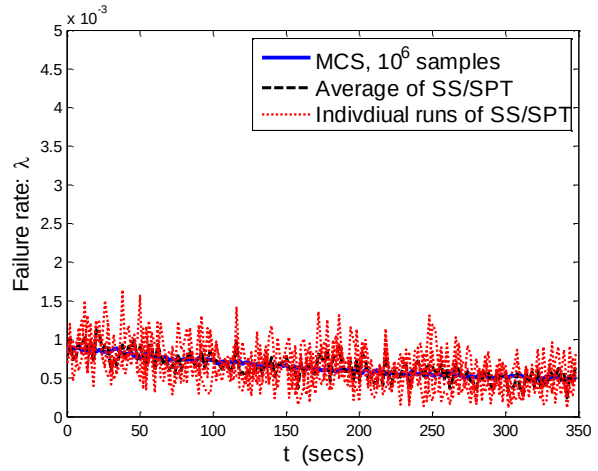
Figure 6 and Table 1 show the calculated failure rate and CDF with a step of 2 seconds. Four different runs were performed (red dotted lines) in order to demonstrate the expected variability. Table 1 provides detailed results for the four independent runs, their mean and the C.O.V. It also shows the MCS estimates. The black dotted line represents the average of the four runs. The estimated probabilities from MCS with 10^6 trajectories (blue solid line) are used as a baseline. Figure 6 shows that the average failure rate and CDF of the four SS/SPT runs is close to the MCS estimates. Figures 7 and 8, and the corresponding Tables 2 and 3, show the calculated failure rate and CDF with a step of 4 and 8 seconds, respectively.

Figures 6a, 7a and 8a show that the estimated failure rate $\lambda(t)$ oscillates around its mean throughout time. The variability is due to the low value of $\lambda(t)$ and the relatively low number of simulated trajectories. However, the estimation is unbiased since the oscillations are about the mean value. The unbiased estimation of the SS/SPT algorithm is also supported by the estimated CDF from the four runs in Figures 6b, 7b and 8b which show variation around their mean. The CDFs do not exhibit an oscillation because the number of parent sample functions N_{sn} was estimated based on the low C.O.V. of 0.07 for each time interval. The probability variation is expected to be small because the increase in the probability of failure P_{cfn} at each time interval is small (approximately equal to $\frac{0.2}{176} = 0.001136$ for the 2 second interval case) and its COV $\delta_{P_{cfn}}$ is chosen small. This is an advantage of the proposed SS/SPT approach over the direct SS/S approach where the estimated CDF can exhibit large oscillations.

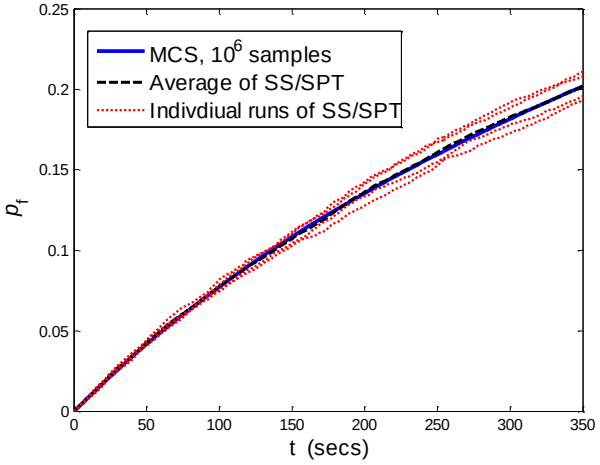
Figure 9 shows the COV of the estimated failure rate $\lambda(t)$ and probability of failure P_f. We observe that the COV of $\lambda(t)$ decreases on average, for increasing time interval (0.3364 for 2 sec, to 0.2459 for 4 sec, to 0.1876 for 8 sec). This is because the error of probability of failure estimation for any MCS-based method reduces as the probability of failure increases. As we increase the time interval in the proposed SS/SPT method, the probability of failure contribution from the larger time interval to the overall P_f increases, reducing therefore the estimation error.

The same observation holds for the C.O.V. of CDF. Figure 9b shows that the C.O.V. decreases on average, for increasing time interval (0.048 for 2 sec, to 0.0314 for 4 sec, to 0.0286 for 8 sec). The average C.O.V. of the CDF is much smaller than the C.O.V. of $\lambda(t)$ due to the higher value of P_f compared to the value of $\lambda(t)$. Furthermore, the C.O.V. of the CDF decreases rapidly in the early period and exhibits less variation at later times because the CDF value increases with increasing time.

Based on the above discussion, a relatively large time interval (e.g. 8 versus 2 seconds for this example) can be used in order to reduce the failure rate and CDF variation.

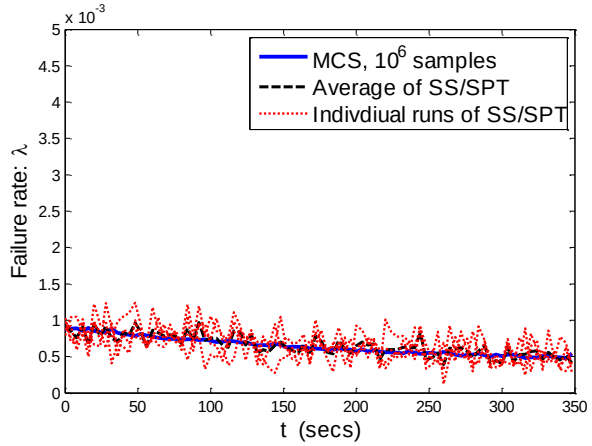


(a)

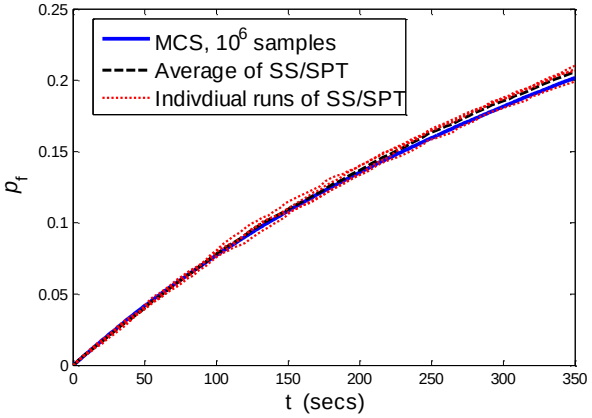


(b)

Figure 6. Failure rate (a) and probability of failure (b) from SS/SPT method calculated at 2 sec intervals

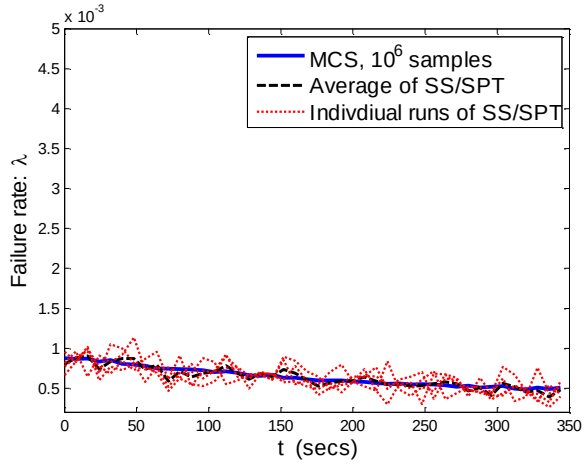


(a)

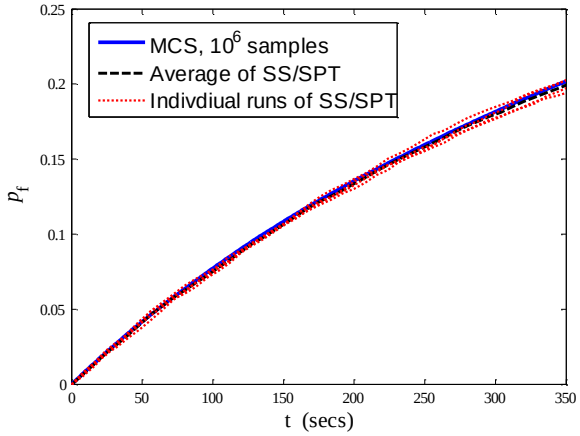


(b)

Figure 7. Failure rate (a) and probability of failure (b) from SS/SPT method calculated at 4 sec intervals

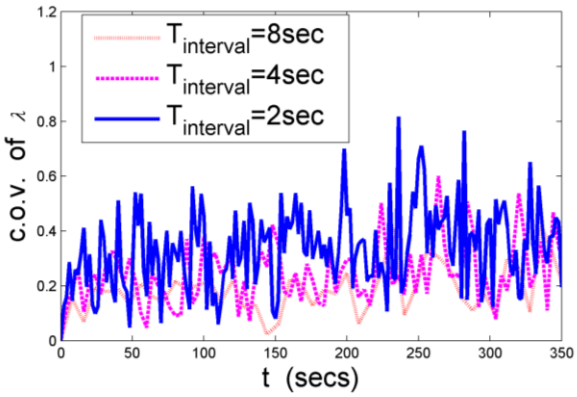


(a)

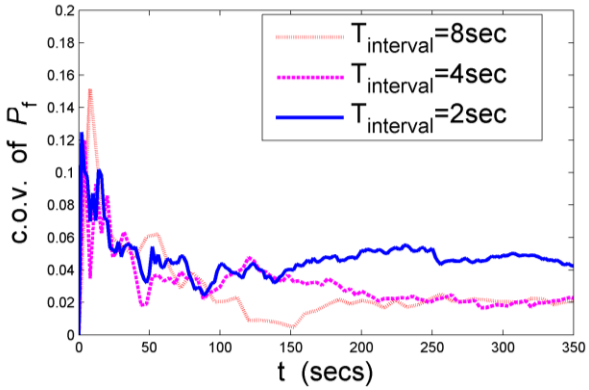


(b)

Figure 8. Failure rate (a) and probability of failure (b) from SS/SPT method calculated at 8 sec intervals



(a)



(b)

Figure 9. C.O.V. of failure rate (a) and probability of failure (b) from SS/SPT method for 2, 4, and 8 sec intervals

Table 1a. Details of from SS/SPT method calculated at 2 sec intervals

Time (sec)	SS/SPT						MCS
	Run 1	Run 2	Run 3	Run 4	Mean	C.O.V.	
2	6.93E-04	7.04E-04	8.47E-04	8.84E-04	7.82E-04	0.124953	8.38E-04
4	8.24E-04	1.20E-03	1.02E-03	9.51E-04	1.00E-03	0.158611	9.12E-04
6	7.83E-04	4.62E-04	9.77E-04	7.33E-04	7.39E-04	0.287182	8.93E-04
8	9.38E-04	6.69E-04	7.49E-04	7.56E-04	7.78E-04	0.146443	8.79E-04
10	1.06E-03	9.83E-04	1.15E-03	5.93E-04	9.48E-04	0.260010	9.24E-04
12	9.23E-04	1.25E-03	7.96E-04	7.56E-04	9.31E-04	0.240193	8.53E-04
14	8.47E-04	8.98E-04	1.50E-03	8.39E-04	1.02E-03	0.315475	8.77E-04
…	…	…	…	…	…	…	…
338	6.30E-04	5.12E-04	4.45E-04	9.05E-04	6.23E-04	0.325363	5.10E-04
340	5.99E-04	2.88E-04	5.33E-04	4.83E-04	4.76E-04	0.282164	4.73E-04
342	4.37E-04	4.68E-04	8.69E-04	4.35E-04	5.52E-04	0.383182	4.87E-04
344	2.78E-04	1.14E-04	3.31E-04	2.62E-04	2.46E-04	0.378173	4.94E-04
346	4.30E-04	3.76E-04	4.81E-04	9.16E-04	5.51E-04	0.449211	4.69E-04
348	7.50E-04	2.81E-04	4.18E-04	4.08E-04	4.64E-04	0.431984	4.81E-04
350	5.53E-04	5.66E-04	8.22E-04	7.32E-04	6.68E-04	0.196244	5.05E-04

Table 1b. Details of P_F from SS/SPT method calculated at 2 sec intervals

Time (sec)	SS/SPT						MCS
	Run 1	Run 2	Run 3	Run 4	Mean	C.O.V.	
2	0.001385	0.001408	0.001694	0.001767	0.001564	0.124953	0.001676
4	0.003031	0.003813	0.00374	0.003666	0.003563	0.100867	0.003497
6	0.004592	0.004735	0.005686	0.005126	0.005035	0.097234	0.005276
8	0.00646	0.006066	0.007176	0.00663	0.006583	0.069924	0.007024
10	0.008575	0.008021	0.009461	0.007807	0.008466	0.087164	0.008859
12	0.010405	0.010498	0.011038	0.009307	0.010312	0.070385	0.010549
14	0.012081	0.012275	0.014012	0.010969	0.012334	0.101987	0.012285
…	…	…	…	…	…	…	…
338	0.191062	0.204623	0.188228	0.205901	0.197454	0.046115	0.197184
340	0.192032	0.205081	0.189094	0.206668	0.198219	0.045125	0.197944
342	0.192738	0.205825	0.190503	0.207358	0.199106	0.043766	0.198725
344	0.193187	0.206006	0.191038	0.207773	0.199501	0.043141	0.199516
346	0.19388	0.206603	0.191816	0.209224	0.200381	0.043937	0.200267
348	0.195089	0.207048	0.192491	0.209869	0.201124	0.042822	0.201037
350	0.195979	0.207946	0.193819	0.211026	0.202193	0.042339	0.201844

Table 2a. Details of $\lambda(t)$ from SS/SPT method calculated at 4 sec intervals

Time (sec)	SS/SPT						MCS
	Run 1	Run 2	Run 3	Run 4	Mean	C.O.V.	
4	9.88E-04	7.89E-04	8.75E-04	1.03E-03	9.22E-04	0.119925	8.74E-04
8	6.98E-04	1.04E-03	9.21E-04	7.50E-04	8.52E-04	0.185275	8.85E-04
12	6.85E-04	1.05E-03	5.91E-04	7.01E-04	7.56E-04	0.263440	8.87E-04
16	9.68E-04	8.36E-04	8.21E-04	1.02E-03	9.10E-04	0.106454	8.51E-04
20	6.34E-04	8.40E-04	5.01E-04	6.70E-04	6.61E-04	0.210439	8.92E-04
24	1.22E-03	6.63E-04	9.89E-04	9.58E-04	9.58E-04	0.240326	8.52E-04
28	9.44E-04	6.44E-04	7.14E-04	8.43E-04	7.86E-04	0.169840	8.39E-04
…	…	…	…	…	…	…	…
328	4.48E-04	5.90E-04	7.23E-04	5.65E-04	5.82E-04	0.194335	4.83E-04
332	5.62E-04	5.57E-04	3.18E-04	4.26E-04	4.66E-04	0.250755	5.04E-04
336	4.58E-04	4.88E-04	8.60E-04	4.30E-04	5.59E-04	0.361246	5.13E-04
340	5.66E-04	4.66E-04	4.73E-04	4.44E-04	4.87E-04	0.110792	4.91E-04
344	3.39E-04	7.74E-04	3.30E-04	3.79E-04	4.55E-04	0.468231	4.90E-04
348	6.32E-04	6.91E-04	3.00E-04	4.54E-04	5.19E-04	0.341906	4.75E-04
352	2.79E-04	3.80E-04	4.44E-04	5.41E-04	4.11E-04	0.267505	5.41E-04

Table 2b. Details of P_F from SS/SPT method calculated at 4 sec intervals

Time (sec)	SS/SPT						MCS
	Run 1	Run 2	Run 3	Run 4	Mean	C.O.V.	
4	0.003953	0.003157	0.003502	0.004134	7.82E-04	0.119925	0.003497
8	0.006734	0.007309	0.007173	0.00712	1.00E-03	0.034798	0.007024
12	0.009457	0.011459	0.00952	0.009904	7.39E-04	0.092894	0.010549
16	0.013292	0.014765	0.012772	0.013931	7.78E-04	0.062751	0.013917
20	0.015796	0.018073	0.014752	0.016575	9.48E-04	0.085815	0.017437
24	0.020617	0.020676	0.018651	0.020342	9.31E-04	0.047741	0.020785
28	0.024316	0.023199	0.021455	0.023644	1.02E-03	0.052766	0.024072
…	…	…	…	…	…	…	…
328	0.199072	0.200064	0.198854	0.191648	6.23E-04	0.019640	0.193087
332	0.200872	0.201847	0.199874	0.193026	4.76E-04	0.020115	0.194714
336	0.202335	0.203406	0.202626	0.194415	5.52E-04	0.020983	0.196365
340	0.20414	0.20489	0.204133	0.195845	2.46E-04	0.021192	0.197944
344	0.205219	0.207351	0.205185	0.197063	5.51E-04	0.022297	0.199516
348	0.20723	0.209541	0.206139	0.19852	4.64E-04	0.023247	0.201037
352	0.208115	0.210742	0.20755	0.200253	6.68E-04	0.021751	0.202766

Table 3a. Details of $\lambda(t)$ from SS/SPT method calculated at 8 sec intervals

Time (sec)	SS/SPT						MCS
	Run 1	Run 2	Run 3	Run 4	Mean	C.O.V.	
8	8.14E-04	9.55E-04	7.70E-04	6.62E-04	8.00E-04	0.151619	8.78E-04
16	9.61E-04	8.30E-04	8.92E-04	8.35E-04	8.79E-04	0.069234	8.68E-04
24	6.45E-04	9.96E-04	9.61E-04	1.02E-03	9.05E-04	0.193195	8.71E-04
32	9.09E-04	7.73E-04	6.86E-04	5.98E-04	7.42E-04	0.178653	8.35E-04
40	8.25E-04	6.81E-04	1.05E-03	7.74E-04	8.33E-04	0.188349	8.51E-04
48	9.20E-04	8.44E-04	9.98E-04	7.25E-04	8.72E-04	0.133392	8.05E-04
56	1.14E-03	7.31E-04	8.13E-04	7.98E-04	8.70E-04	0.207804	7.97E-04
…	…	…	…	…	…	…	…
304	4.09E-04	4.38E-04	7.67E-04	6.22E-04	5.59E-04	0.091557	5.28E-04
312	5.39E-04	5.21E-04	6.71E-04	3.39E-04	5.17E-04	0.300373	4.93E-04
320	4.43E-04	5.03E-04	4.66E-04	3.90E-04	4.51E-04	0.263403	4.86E-04
328	4.28E-04	3.52E-04	3.38E-04	7.57E-04	4.69E-04	0.104947	5.08E-04
336	3.25E-04	2.65E-04	5.42E-04	4.32E-04	3.91E-04	0.418887	4.90E-04
344	5.83E-04	3.97E-04	4.21E-04	5.17E-04	4.79E-04	0.311981	5.08E-04
352	4.09E-04	4.38E-04	7.67E-04	6.22E-04	5.59E-04	0.179697	5.28E-04

Table 3b. Details of P_F from SS/SPT method calculated at 8 sec intervals

Time (sec)	SS/SPT						MCS
	Run 1	Run 2	Run 3	Run 4	Mean	C.O.V.	
8	0.00652	0.00764	0.00616	0.00529	0.006402	0.151619	0.007024
16	0.01415	0.01423	0.01325	0.01194	0.013393	0.079600	0.013917
24	0.01924	0.02208	0.02084	0.01998	0.020535	0.059432	0.020785
32	0.02637	0.02813	0.02621	0.02467	0.026347	0.053640	0.027326
40	0.03280	0.03343	0.03439	0.03071	0.032834	0.047451	0.033949
48	0.03992	0.03995	0.04211	0.03634	0.03958	0.060412	0.040174
56	0.04864	0.04557	0.04834	0.04249	0.04626	0.061983	0.046292
…	…	…	…	…	…	…	…
304	0.18609	0.17862	0.18231	0.17817	0.181296	0.020390	0.183273
312	0.18875	0.18150	0.18733	0.18226	0.184959	0.019571	0.186726
320	0.19225	0.18491	0.19169	0.18448	0.188331	0.022360	0.189936
328	0.19511	0.18819	0.19470	0.18702	0.191256	0.022200	0.193087
336	0.19787	0.19048	0.19688	0.19195	0.194292	0.018684	0.196365
344	0.19995	0.19219	0.20036	0.19474	0.196812	0.020341	0.199516
352	0.20368	0.19476	0.20306	0.19807	0.199891	0.021240	0.202766

5. SUMMARY, CONCLUSIONS AND FUTURE WORK

We presented an improved subset simulation with splitting approach to calculate the time-dependent probability of failure for dynamic systems with uncertain parameters subjected to stochastic excitation. The method partitions the high dimensional random process of the response into a series of correlated, short duration, low dimensional random processes. Subset simulation reduces the computational cost by introducing appropriate intermediate failure sub-domains to express the low failure probability as a product of larger conditional failure probabilities. Splitting provides an efficient sampling scheme to estimate the conditional probabilities. The proposed subset simulation with splitting not only estimates the time-dependent probability of failure at a given time but also estimates the failure rate and cumulative distribution function up to that time with approximately the same cost. We used the vibratory response of a vehicle over a stochastic terrain to demonstrate the accuracy and efficiency of the proposed approach.

In future work, we plan to enhance the developed method by using splitting not only in the threshold direction but also in the time direction.

ACKNOWLEDGMENT

We would like to acknowledge the technical and financial support of the Automotive Research Center (ARC) in accordance with Cooperative Agreement W56HZV-04-2-0001 U.S. Army Tank Automotive Research, Development and Engineering Center (TARDEC)

REFERENCES

1. Savage, G. J., and Son, Y. K., "Dependability-Based Design Optimization of Degrading Engineering Systems," *ASME Journal of Mechanical Design*, 131(1), 011002 (10 pages), 2009.

2. Ascher, H., and Feingold, H., "Repairable Systems Reliability", *Marcel Dekker*, New York, N.Y., 1984.

3. Singh, A., Mourelatos, Z. P., and Li, J., "Design for Lifecycle Cost using Time-dependent Reliability," *ASME Journal of Mechanical Design*, 132(9), 091008 (11 pages), 2010. Also, *Proceedings of ASME 2009 Design Engineering Technical Conferences*, Paper DETC2009-86587/DAC-29, San Diego, CA, August 2009.

4. Son, Y. K., Chang, S. W., and Savage, G. J., "Economic-based Design of Engineering Systems with Degrading Components Using Probabilistic Loss of Quality," *Journal of Mechanical Science and Technology*, 21(2), 225-234, 2007.

5. Singh, A., Mourelatos, Z. P., and Li, J., "Design for Lifecycle Cost and Preventive Maintenance Using Time-Dependent Reliability," *Advanced Materials Research*, Vols. 118-120, 10-16, 2010.

6. Andrieu-Renaud, C., Sudret, B., and Lemaire, M., "The PHI2 Method: A Way to Compute Time-Variant Reliability", *Reliability Engineering and Safety System*, 84(1), 75-86, 2004.

7. Hagen, O., and Tvedt, L., "Vector Process Out-Crossing as Parallel System Sensitivity Measure," *ASCE Journal of Engineering Mechanics*, 121(10), 2201-2210, 1991.

8. Rice, S. O., "Mathematical Analysis of Random Noise," *Bell Syst Tech J*, 23, 282–332, 1954. [Re-published in: Wax N, editor. *Selected Papers on Noise and Stochastic Processes*, New York, Dover, 1954].

9. Rackwitz, R., "Computational Techniques in Stationary and Non-Stationary Load Combination – A Review and Some Extensions," *Journal of Structural Engineering*, 25(1), 1-20, 1998.

10. Schueller, G. I., "A State-of-the-art Report on Computational Stochastic Mechanics," *Probabilistic Engineering Mechanics*, 12(4), 197-321, 1997.

11. Engelung, S., Rackwitz, R., and Lange, C., "Approximations of First Passage Times for Differentiable Processes Based on Higher Order Threshold Crossings," *Probabilistic Engineering Mechanics*, 10(1), 53-60, 1995.

12. Sudret, B., "Analytical Derivation of the Outcrossing Rate in Time-variant Reliability Problems," *Structure and Infrastructure Engineering*, 4(5), 356-362, 2008.

13. Zhang, J., and Du, X., "Time-dependent Reliability Analysis for Function Generator Mechanisms," *ASME Journal of Mechanical Design*, 133(3), 031005 (9 pages), 2011.

14. Shinozuka, M., "Probability of Failure under Random Loading," *ASCE Journal of Engineering Mechanics*, 90, 147-171, 1964.

15. Son, Y. K., and Savage, G. J., "Set Theoretic Formulation of Performance Reliability of Multiple Response Time-Variant Systems due to Degradations in System Components," *Quality and Reliability Engineering International*, 23(2), 171-188, 2007.

16. Singh, A., and Mourelatos, Z. P., "On the Time-dependent Reliability of Non-Monotonic, Non-Repairable Systems," *SAE International Journal of Materials and Manufacturing*, 3(1), 425-444, 2010.

17. Hu, Z., and Du, X., "Time-dependent Reliability Analysis by a Sampling Approach to Extreme Values of

Stochastic Processes," *Proceedings of ASME 2012 Design Engineering Technical Conferences*, Paper DETC2012-70132, Chicago, IL, 2012.

18. Hu, Z., Li, H., Du, X., and Chandrashekhara, K., "Simulation-based Time-dependent Reliability Analysis for Composite Hydrokinetic Turbine Blades," *Structural and Multidisciplinary Optimization*, DOI 10.1007/s00158-012-0839-8, 2012.

19. Li, J., and Mourelatos, Z. P., "Time-Dependent Reliability Estimation for Dynamic Problems using a Niching Genetic Algorithm," *ASME Journal of Mechanical Design*, 131(7), 2009.

20. Hu, Z., and Du, X., "Reliability Analysis for Hydrokinetic Turbine Blades," *Renewable Energy*, 48, 251-262, 2012.

21. Madsen, P. H., and Krenk, S., "An Integral Equation Method for the First Passage Problem in Random Vibration," *Journal of Applied Mechanics*, 51, 674-679, 1984.

22. Melchers, R. E., "Importance Sampling in Structural Systems," *Structural Safety*, 6, 3-10, 1989.

23. Au, S. K., and Beck, J. L., "A New Adaptive Importance Sampling Scheme," *Structural Safety*, 21, 135-158, 1999.

24. A. Singh and Z.P. Mourelatos, "Time-Dependent Reliability Estimation for Dynamic Systems Using a Random Process Approach," *SAE International Journal of Materials & Manufacturing*, 3(1), 339-355, 2010.

25. A. Singh, Z.P. Mourelatos and E. Nikolaidis, "Time-Dependent Reliability of Random Dynamic Systems using Time-Series Modeling and Importance Sampling," *SAE International Journal of Materials & Manufacturing*, 4(1), 929-946, 2011.

26. Au, S. K., and Beck, J. L., "Estimation of Small Failure Probability in High Dimensions Simulation," *Probabilistic Engineering Mechanics*, 16, 263-277, 2001.

27. Au, S. K., and Beck, J. L., "Subset Simulation and its Application to Seismic Risk Based on Dynamic Analysis," *Journal of Engineering Mechanics*, 129, 901-917, 2003.

28. Norouzi, M., and Nikolaidis, E., "Efficient Random Vibration Analysis Using Markov Chain Monte Carlo Simulation," *SAE International Journal of Materials & Manufacturing*, 5(1), 77-86, 2012.

29. Beck, J. L., and Au, S. K., "Bayesian Updating of Structural Models and Reliability Using Markov Chain Monte Carlo Simulation," *Journal of Engineering Mechanics*, 128(4), 380-391, 2002.

30. Ching, J., Beck, J. L., and Au, S. K., "Hybrid Subset Simulation Method for Reliability Estimation of Dynamic Systems Subject to Stochastic Excitation," *Structural Safety*, 20(3), 199-214, 2005.

31. Ching, J., Beck, J. L., and Au, S. K., "Reliability Estimation for Dynamic Systems Subject to Stochastic Excitation Using Subset Simulation with Splitting," *Computer Methods in Applied Mechanics and Engineering*, 194, 1557-1579, 2005.

32. Morio, J., Pastel, R., and Gland, F., "An Overview of Importance Splitting for Rare Event Simulation," *European Journal of Physics*, 31(5), 1295-1303, 2010.

33. Rao, S. S., *Mechanical Vibrations*, 5th edition, Prentice Hall, 2011.

34. Ho, S., and Xie, M., "The Use of ARIMA Models for Reliability Forecasting and Analysis", *Computers in Engineering*, 35(1-2), 213-216, 1998.

35. Ruppert, D., *Statistics in Finance*, Springer-Verlag, New York, 2004.